

STABLE COMPLETE EMBEDDED MINIMAL SURFACES IN \mathbb{H}^1 WITH EMPTY CHARACTERISTIC LOCUS ARE VERTICAL PLANES

D. DANIELLI, N. GAROFALO, D. M. NHIEU, AND S. D. PAULS

ABSTRACT. In the recent paper [12] we have proved that the only stable C^2 minimal surfaces in the first Heisenberg group \mathbb{H}^n which are graphs over some plane and have empty characteristic locus must be vertical planes. This result represents a sub-Riemannian version of the celebrated theorem of Bernstein.

In this paper we extend the result in [12] to C^2 complete embedded minimal surfaces in \mathbb{H}^1 with empty characteristic locus. We prove that every such a surface without boundary must be a vertical plane.

1. INTRODUCTION

The study of minimal surfaces has been one of the prime drivers of the study of geometry and calculus of variations in the twentieth century and, in particular, the Bernstein problem has played a central role. Bernstein proved his Theorem [4], that a C^2 minimal graph in \mathbb{R}^3 must necessarily be an affine plane in 1915 and, almost fifty years later, a new insight of Fleming [16] generated renewed interest in the problem. The work of De Giorgi, [13], Almgren, [1], Simons, [28], and Bomberli-De Giorgi-Giusti, [5], culminated in the complete solution to the Bernstein problem:

Theorem 1.1. *Let $S = \{(x, u(x)) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n, x_{n+1} = u(x)\}$ be a C^2 minimal graph in \mathbb{R}^{n+1} , i.e., let $u \in C^2(\mathbb{R}^n)$ be a solution of the minimal surface equation*

$$(1) \quad \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0,$$

in the whole space. If $n \leq 7$, then there exist $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ such that $u(x) = \langle a, x \rangle + \beta$, i.e., S must be an affine hyperplane. If instead $n \geq 8$, then there exist non affine (real analytic) functions on \mathbb{R}^n which solve (1).

Roughly a decade later, Fischer-Colbrie and Schoen, [15], and do Carmo and Peng, [14], imposing a stability condition, independently proved a far reaching generalization of the Bernstein property:

Theorem 1.2. *Every stable complete minimal surface $S \subset \mathbb{R}^3$ must be a plane.*

Here, stable means that on every compact set S minimizes area up to order two. We note in passing that, thanks to the strict convexity of the area functional $\mathcal{A}(u) = \int_{\Omega} \sqrt{1 + |Du|^2} dx$, where $\Omega \subset \subset \mathbb{R}^n$, for Euclidean graphs on \mathbb{R}^n the stability assumption is automatically satisfied.

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The purpose of this paper is to prove an analogue of Theorem 1.2 in the sub-Riemannian Heisenberg group \mathbb{H}^1 (for the relevant definitions we refer the reader to the next section). The study of the Bernstein problem in this setting has received increasing attention over the last decade. The existence of minimal surfaces in sub-Riemannian spaces was established by two of us in [19] by developing in such setting the methods of the geometric measure theory. The study of minimal graphs in the Heisenberg group was first approached by one of us in [25], by Cheng, Hwang, Malchiodi and Yang [8] (who studied the problem in a more general class of pseudohermitian 3-manifolds), by three of us in [11], and by two of us in [20].

Henceforth in this paper, following a perhaps unfortunate but old tradition, by *minimal* we intend a C^2 surface $S \subset \mathbb{H}^1$ whose sub-Riemannian, or horizontal mean curvature \mathcal{H} (see Proposition 2.3 below for its expression) vanishes identically on S . In these initial investigations, a number of nonplanar minimal graphs over the xy -plane are produced ([25, 8, 20]) and indeed are classified (first in [8], with an alternate proof in [20]). A prototypical example is given by the surface $t = xy/2$ which is an entire minimal graph over the xy -plane. However, this example and all other entire minimal graphs over the xy -plane must have non empty characteristic locus (this fact was proved independently in [8] and [20]). We recall that the latter is defined as the set of points of the surface at which the two bracket generating vector fields X_1, X_2 become tangent to the surface itself.

In some of these same papers, new examples were discovered of entire minimal graphs over some plane, but with an empty characteristic locus. In [20], two of us first produced infinitely many examples of such graphs, one of which is given by

$$(2) \quad x = y \tan(\tanh(t)).$$

Moreover, as announced in [8] (this and many other examples are shown in more detail in [7]), the surface

$$(3) \quad x = y t$$

is also noncharacteristic and minimal. From the point of view of the Bernstein problem, these examples would indicate a failure of the property - there exists a rich reservoir of graphs over the xy -plane which are minimal (although they have characteristic points) and an equally rich reservoir of nonplanar noncharacteristic minimal graphs over the yt -plane (or the xt -plane). In the positive direction, the work [20] shows that graphs over vertical planes must have a specific structure indicating some kind of rigidity (see also [7] for other classification results).

In [10] the first three authors continued the investigation into noncharacteristic graphs by asking a more refined question: are surfaces such as (2) or (3) local minima? Just as in the classical case, sub-Riemannian minimal surfaces are shown to merely be critical points of the relative area functional (the so-called horizontal perimeter). Since this functional is shown to lack the fundamental convexity property which guarantees in the flat case that critical points are global minimizers, the question of stability becomes central. It could happen in fact that minimal surfaces such as (2), (3) might fail to be locally area minimizing. Using a basic second variation formula discovered in [11], in [10] the following surprising theorem is proved.

Theorem 1.3. *Let $\alpha > 0, \beta \in \mathbb{R}$, then the surfaces*

$$x = y (\alpha t + \beta), \quad y = x (-\alpha t + \beta),$$

are unstable noncharacteristic entire minimal graphs.

We emphasize that these surfaces are also global intrinsic graphs in the sense of [17], [18], see Definition 1.7 below. We also note that Theorem 1.3 shows that an analogue of the Bernstein property cannot hold unless we assume the surface be noncharacteristic and stable.

The second variation formula in [11] reduces to a stability inequality of Hardy type on the surface. Another major tool in the proof of Theorem 1.3 is the reduction of such Hardy type inequality to a one dimensional integral inequality of Carleman-Wirtinger type which is confirmed by explicitly constructing a variation which decreases perimeter. In [12], we continued this line of investigation and provided a positive answer to the following version of the Bernstein problem.

Theorem 1.4 (Bernstein Theorem 1, [12]). *In \mathbb{H}^1 the only stable C^2 minimal entire graphs, with empty characteristic locus, are the vertical planes (6).*

To illustrate the strategy behind this result, we recall a definition from [12].

Definition 1.5. *We say that a C^2 surface $S \subset \mathbb{H}^1$ is a graphical strip if there exist an interval $I \subset \mathbb{R}$, and $G \in C^2(I)$, with $G' \geq 0$ on I , such that, after possibly a left-translation and a rotation about the t -axis, then either*

$$(4) \quad S = \{(x, y, t) \in \mathbb{H}^1 \mid (y, t) \in \mathbb{R} \times I, x = yG(t)\},$$

or

$$(5) \quad S = \{(x, y, t) \in \mathbb{H}^1 \mid (x, t) \in \mathbb{R} \times I, y = -xG(t)\}.$$

If there exists $J \subset I$ such that $G' > 0$ on J , then we call S a strict graphical strip.

It should be immediately clear to the reader that the surfaces in (2) or (3) are examples of strict graphical strips in which one can take $J = I = \mathbb{R}$. Here is one of the two central results of [12].

Theorem 1.6. *Any strict graphical strip is an unstable minimal surface with empty characteristic locus. As a consequence, any minimal surface containing a strict graphical strip is unstable.*

The proof of Theorem 1.6 involves, among other things, an adaptation of the technique in [10] which leads to the construction of an explicit variation along which the horizontal perimeter strictly decreases. Our second main result in [12] consists in proving, using the techniques of [20], that every noncharacteristic minimal graph over some plane which is not itself a vertical plane

$$(6) \quad \Pi_\gamma = \{(x, y, t) \in \mathbb{H}^1 \mid ax + by = \gamma\},$$

contains a strict graphical strip. Combining this result with Theorem 1.6, we obtain Theorem 1.4.

Another approach to the sub-Riemannian Bernstein problem arises when considering an intrinsic notion of graph. Observe that in flat \mathbb{R}^3 a graph of the type $S = \{x = \phi(y, z) \mid (y, z) \in \Omega\}$, can be written as $S = \{(0, u, v) + \phi(u, v)e_1 \mid (u, v) \in \Omega\}$, where we have let $e_1 = (1, 0, 0)$. Inspired by this observation Franchi, Serapioni and Serra Cassano proposed the following notion of intrinsic graph adapted to the non-Abelian group structure of \mathbb{H}^1 .

Definition 1.7. *A C^2 surface S is an intrinsic X_1 -graph if there exist a domain $\Omega \subset \mathbb{R}_{uv}^2$ and $\phi \in C^2(\Omega)$, such that $S = \{(0, u, v) \circ \phi(u, v)e_1 \mid (u, v) \in \Omega\}$.*

We note that one basic consequence of this definition is that S has empty characteristic locus. This follows from the fact that the vector field X_1 is always transversal to the surface. Interestingly, if we assume that Ω be bounded, then the horizontal perimeter of S is given by the formula

$$(7) \quad \mathcal{P}_H(S) = \int_\Omega \sqrt{1 + \mathcal{B}_\phi(\phi)^2} \, du \, dv,$$

where we have denoted by $\mathcal{B}_\phi(f) = f_u + \phi f_v$ the linearized Burger's operator. Notice that $\mathcal{B}_\phi(\phi) = \phi_u + \phi\phi_v$ is the nonlinear inviscid Burger's operator. Definition 1.7 was first introduced in [17] and developed further in [18, 2, 3, 21]. In [3], Barone Adesi, Serra Cassano and Vittone prove the following Bernstein theorem for these types of graphs.

Theorem 1.8 (Bernstein Theorem 2, [3]). *The only C^2 , stable minimal entire intrinsic X_1 -graphs are the vertical planes.*

The proof of Theorem 1.8 relies on a clever choice of coordinates, suggested by the study of the characteristic curves of the solutions of the minimal surface equation, which for an intrinsic graph becomes

$$(8) \quad \mathcal{B}_\phi(\mathcal{B}_\phi(\phi)) = 0.$$

Such change of coordinates allows the authors to reduce to a case which can again be solved using the one dimensional reduction techniques used in [10] to prove Theorem 1.3.

We are now in a position to discuss the results of this paper. First, we introduce a definition which is related to Definition 1.5 and that is suggested by the analysis of the double Burger equation (8). Suppose we are given some interval $J = (-4\epsilon, 4\epsilon) \subset \mathbb{R}$, $\epsilon > 0$, and functions $F, G, \sigma \in C^2(J)$ satisfying the condition

$$(9) \quad F'(s)^2 < 2\sigma'(s)G'(s), \quad \text{for every } s \in J.$$

We note explicitly that (9) implies, in particular, that $\sigma'(s)G'(s) > 0$ for every $s \in J$. If we consider the mapping $\Psi : \mathbb{R} \times J \rightarrow \mathbb{R}^2$ from the (u, s) to the (u, v) plane defined by $\Psi(u, s) = (u, v)$, where v is defined by the equation

$$(10) \quad v = v(u, s) = G(s)\frac{u^2}{2} + F(s)u + \sigma(s),$$

then we see that the Jacobian determinant of Ψ is given by

$$(11) \quad \begin{aligned} \det J_\Psi(u, s) &= \det \begin{pmatrix} 1 & 0 \\ G(s)u + F(s) & G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s) \end{pmatrix} \\ &= G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s). \end{aligned}$$

Thanks to (9) the Jacobian of Ψ is always different from zero. We emphasize at this moment that the continuity of the first derivatives of the functions F, G and σ , along with the assumption (9), guarantee that, possibly restricting the interval $J = (-4\epsilon, 4\epsilon)$, we can always force the map Ψ to be globally one-to-one, hence a C^2 diffeomorphism of $\mathbb{R} \times J$ onto its image $\Psi(\mathbb{R} \times J)$. We denote with $\Psi^{-1}(u, v) = (u, s(u, v))$ the inverse C^2 diffeomorphism. When we write $s(u, v)$ we mean the function specified by such inverse diffeomorphism.

Definition 1.9. *Let $\epsilon > 0$, $J = (-4\epsilon, 4\epsilon)$. A C^2 surface $S \subset \mathbb{H}^1$ is an intrinsic graphical strip on J if there exist functions $F, G, \sigma \in C^2(J)$ satisfying $(F')^2 \leq 2\sigma'G'$ such that, if*

$$\Omega = \Psi(\mathbb{R} \times J) = \{(u, v) | u \in \mathbb{R}, v = G(s)\frac{u^2}{2} + F(s)u + \sigma(s) \text{ for } s \in J\},$$

then with $\phi \in C^2(\Omega)$ defined by

$$\phi(u, v) = F(s(u, v)) + uG(s(u, v)),$$

we have

$$S = \{(0, u, v) \circ (\phi(u, v), 0, 0) | (u, v) \in \Omega\} = \{(\phi(u, v), u, v - \frac{u}{2}\phi(u, v)) | (u, v) \in \Omega\}.$$

We say that S is a strict intrinsic graphical strip on J if F, G, σ satisfy the strict inequality (9), and if the map $\Psi : \mathbb{R} \times J \rightarrow \Omega$ is globally one-to-one, hence a diffeomorphism of $\mathbb{R} \times J$ onto $\Psi(\mathbb{R} \times J) = \Omega$.

Remark 1.10. A strict intrinsic graphical strip is necessarily a minimal surface. To see this, we observe that the function ϕ in the above definition satisfies (8). The reader will find most of the computations to achieve this in the proof of Corollary 3.6, see formula (23) below, and the computations following that formula.

Remark 1.11. In the case of a strict intrinsic graphical strip, without loss of generality we can assume that $G'(s) > 0$ on J (this property is needed in the proof of Lemmas 4.1, 4.2 and Theorem A). We can justify this claim as follows. As observed earlier, the condition (9) implies $\sigma'(s)G'(s) > 0$. Since $\sigma'(s)$ does not change sign, if $\sigma'(s) > 0$, this forces $G'(s) > 0$. If instead we have $G'(s) < 0$ on J , we replace F, G, σ by $\tilde{F}(s) = F(-s)$, $\tilde{G}(s) = G(-s)$, $\tilde{\sigma}(s) = \sigma(-s)$. The newly defined functions satisfy (9). We also have $\tilde{G}'(s) > 0$. Now we take $\phi(u, v) = \tilde{F}(-s(u, v)) + u\tilde{G}(-s(u, v))$. We see that the surface parameterized by this new ϕ has the same trace as the one with the original ϕ .

Remark 1.12. We emphasize here that any vertical plane such as (6) is an intrinsic graphical strip, but not a strict intrinsic graphical strip. One has in fact if $a \neq 0$, that $\phi(u, v) = \frac{\gamma}{a} - \frac{b}{a}u$, so that $F(s) \equiv \frac{\gamma}{a}$, $G(s) = -\frac{b}{a}$, $\sigma(s) \equiv 0$. Therefore, $2\sigma'G' - (F')^2 \equiv 0$.

Notice that, as a consequence of the smoothness hypothesis on F, G , an intrinsic graphical strip is a surface of class C^2 . Definition 1.9 takes advantage of the coordinates introduced in [3] discussed above, and the motivation behind it will be explained in Section 5. With this definition in place and the second variation formula written in terms of intrinsic graphical strips, in Section 4 we use techniques from [10] (and modifications from [12]) to construct a variation on an intrinsic graphical strip which decreases the horizontal perimeter, proving the following basic result.

Theorem A. Let S be a strict intrinsic graphical strip as in Definition 1.9. There exists a $\psi \in C_0^2(S)$ such that

$$\mathcal{V}_{II}^H(S, \psi X_1) < 0.$$

As a consequence, S is unstable.

The relevance of Theorem A is in the following theorem, which we prove in Section 5.

Theorem B. Every C^2 complete noncompact embedded minimal surface without boundary with empty characteristic locus and which is not itself a vertical plane contains a strict intrinsic graphical strip.

Our proof of Theorem B hinges on a close analysis of the representation results of [20]. Theorems A and B are the main novel technical points of the present paper. From them, the following theorem of Bernstein type will follow.

Theorem C (of Bernstein type). The vertical planes are the only stable C^2 complete embedded minimal surfaces in \mathbb{H}^1 without boundary and with empty characteristic locus.

We note that Theorem C is not contained in either of the cited Theorems 1.4 or 1.8. For instance the sub-Riemannian *catenoids* in \mathbb{H}^1 (the reader should note that these surfaces are just the classical hyperboloids of revolution)

$$(12) \quad (t - a)^2 = \frac{4}{b^2} \left(\frac{b}{4}(x^2 + y^2) - 1 \right), \quad a, b \in \mathbb{R}, b > 0,$$

are complete embedded minimal surfaces with empty characteristic locus which are not graphs on any plane, nor they are entire intrinsic graphs. Theorem C shows that such minimal surfaces are unstable. These surfaces are a model of special interest. For this reason, and also for making transparent to the reader our more general constructions, we discuss them in detail in section 5.1.

In closing, we note that the representation results of this paper require that the surface be C^2 : the complete regularity theory of minimal surfaces is currently an open problem which is being very actively investigated.

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2. DEFINITIONS

In this section we recall some definitions and known results which will be needed in the paper. We recall that the Heisenberg group \mathbb{H}^n is the graded, nilpotent Lie group of step 2 with underlying manifold is $\mathbb{C}^n \times \mathbb{R} \cong \mathbb{R}^{2n+1}$, whose points we indicate $g = (x, y, t)$, $g' = (x', y', t')$, etc., with non-Abelian left-translation

$$(13) \quad L_g(g') = g \circ g' = \left(x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - x' \cdot y) \right),$$

and non-isotropic dilations

$$(14) \quad \delta_\lambda(g) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0.$$

Here, and throughout the paper, we will use $v \cdot w$ to denote the standard Euclidean inner product of two vectors v and w in \mathbb{R}^n . The dilations (14) provide a natural scaling associated with the grading of the Heisenberg algebra $\mathfrak{h}_n = V_1 \oplus V_2$, where $V_1 = \mathbb{R}^{2n} \times \{0\}$, $V_2 = \{0\} \times \mathbb{R}$. According to such scaling, elements of the horizontal layer V_1 have degree one, whereas elements of the vertical layer V_2 are assigned the degree two. The homogeneous dimension associated with (14) is $Q = 2n + 2$. We recall that, identifying \mathfrak{h}_n with \mathbb{R}^{2n+1} , we have for the bracket

$$[g, g'] = (0, 0, x \cdot y' - x' \cdot y).$$

It is then clear that $[V_1, V_1] = V_2$, and that V_2 is the group center.

Henceforth, we will focus on the first Heisenberg group \mathbb{H}^1 . Applying the differential $(L_g)_*$ of (13) to the standard basis $\{\partial_x, \partial_y, \partial_t\}$ of \mathbb{R}^3 , we obtain the three distinguished vector fields

$$X_1 = (L_g)_*(\partial_x) = \partial_x - \frac{y}{2}\partial_t, \quad X_2 = (L_g)_*(\partial_y) = \partial_y + \frac{x}{2}\partial_t, \quad T = (L_g)_*(\partial_t) = \partial_t.$$

The horizontal bundle $H\mathbb{H}^1$ is the subbundle of $T\mathbb{H}^1$ whose fiber at a point $g \in \mathbb{H}^1$ is given by

$$H_g = \text{span}\{X_1(g), X_2(g)\}.$$

We endow \mathbb{H}^1 with a left-invariant Riemannian metric $\{g_{ij}\}$, whose inner product we will denote by $\langle \cdot, \cdot \rangle$, with respect to which $\{X_1, X_2, T\}$ constitute an orthonormal basis. If $S \subset \mathbb{H}^1$ is a C^2 oriented surface we will indicate with N a (non-unit) Riemannian normal with respect to $\langle \cdot, \cdot \rangle$, and with $\nu = N/|N|$ the corresponding Gauss map. We will let

$$(15) \quad p = \langle N, X_1 \rangle, \quad q = \langle N, X_2 \rangle, \quad W = \sqrt{p^2 + q^2}, \quad \omega = \langle N, T \rangle.$$

The characteristic locus of S is the closed subset of S defined by

$$\Sigma(S) = \{g \in S \mid W(g) = 0\}.$$

We notice explicitly that $\Sigma(S) = \{g \in S \mid T_g S = H_g\}$. We also set on $S \setminus \Sigma(S)$

$$(16) \quad \bar{p} = \frac{p}{W}, \quad \bar{q} = \frac{q}{W}, \quad \bar{\omega} = \frac{\omega}{W}.$$

Definition 2.1. Let $S \subset \mathbb{H}^1$ be a C^2 oriented surface. A horizontal normal of S is defined as

$$N^H = pX_1 + qX_2,$$

whereas on $S \setminus \Sigma(S)$ the horizontal Gauss map is defined as

$$\nu^H = \frac{1}{W}N^H = \bar{p}X_1 + \bar{q}X_2.$$

The horizontal perimeter measure of S has the following form.

Proposition 2.2. Let $S \subset \mathbb{H}^1$ be a C^2 oriented surface, then the horizontal perimeter of S is

$$\mathcal{P}_H(S) = \int_S \sqrt{\langle \nu, X_1 \rangle^2 + \langle \nu, X_2 \rangle^2} d\sigma = \int_S \frac{W}{|N|} d\sigma,$$

where $d\sigma$ is the Riemannian surface area element associated to $\langle \cdot, \cdot \rangle$.

To investigate minimal surfaces, we recall the notion of horizontal mean curvature \mathcal{H} introduced in [11], [25], [20]. Such notion is obtained by projecting the horizontal Levi-Civita connection onto the so-called horizontal tangent bundle $HTS = TS \cap H\mathbb{H}^1$. If we assume, as we may, that the Riemannian normal field on S , N^H , can be extended to a neighborhood of S , and continuing to denote by \bar{p}, \bar{q} the quantities introduced in (16) relative to such extension, then it has been shown in the above cited references that \mathcal{H} can be computed by the following proposition.

Proposition 2.3. For $g \in S \setminus \Sigma(S)$, the H -mean curvature of S at g is given by

$$\mathcal{H}(g) = X_1\bar{p}(g) + X_2\bar{q}(g).$$

For $g \in \Sigma(S)$, we define $\mathcal{H}(g) = \lim_{g' \rightarrow g, g' \in S \setminus \Sigma(S)} \mathcal{H}(g')$, whenever the limit exists. A surface S is said to be *minimal* if its horizontal mean curvature vanishes identically.

It is now well known ([8, 26, 20, 11, 25]) that critical points of the perimeter are characterized by having zero H -mean curvature away from the characteristic locus. We mention that recent work of Cheng, Hwang and Yang ([9]) and Ritoré and Rosales ([27]) have clarified the behavior of such critical points at the characteristic locus. However, since we will be restricting to the category of noncharacteristic surfaces, we will not discuss these results here.

3. THE SECOND VARIATION OF THE HORIZONTAL PERIMETER AND THE STABILITY OF MINIMAL SURFACES

In this section, we recall the first and second variation of the horizontal perimeter for intrinsic graphs. We mention that formulas for the first and second variation of the horizontal perimeter have been derived a number of times in various contexts ([26, 27, 8, 11, 21, 2, 3, 22, 23, 6]).

Let $S \subset \mathbb{H}^1$ be an oriented C^2 surface with empty characteristic locus, and consider vector fields $\mathcal{X} = aX_1 + bX_2 + kT$, with $a, b, k \in C_0^2(S)$. We define the *first variation* of the horizontal perimeter with respect to the deformation of S ,

$$S^\lambda = S + \lambda \mathcal{X},$$

as

$$\mathcal{V}_I^H(S; \mathcal{X}) = \frac{d}{d\lambda} P_H(S^\lambda) \Big|_{\lambda=0}.$$

We say that S is *stationary* if $\mathcal{V}_I^H(S; \mathcal{X}) = 0$, for every \mathcal{X} . We define the *second variation* of the horizontal perimeter as

$$\mathcal{V}_{II}^H(S; \mathcal{X}) = \frac{d^2}{d\lambda^2} P_H(S^\lambda) \Big|_{\lambda=0}.$$

We say that S is *stable* if $\mathcal{V}_{II}^H(S; \mathcal{X}) \geq 0$ for every \mathcal{X} .

Henceforth, to simplify the formulas we introduce the following notation

$$(17) \quad F_{\mathcal{X}} \stackrel{\text{def}}{=} \bar{p}a + \bar{q}b + \bar{w}k = \frac{\langle \mathcal{X}, \mathbf{N} \rangle}{\langle \nu^H, \mathbf{N} \rangle}.$$

The following result was proved independently by several people in various contexts, see[26, 27, 8, 11, 21, 2, 3, 22, 23, 6].

Theorem 3.1. *Let $S \subset \mathbb{H}^1$ be an oriented C^2 surface with empty characteristic locus, then*

$$(18) \quad \mathcal{V}_I^H(S; \mathcal{X}) = \int_S \mathcal{H} F_{\mathcal{X}} d\sigma_H.$$

In particular, S is stationary if and only if it is minimal.

To state the next result we introduce a notation. Given the quantity \bar{w} we let

$$\mathcal{A} = -\nabla^{H,S} \bar{w}.$$

The following second variation formula was proved in [11].

Theorem 3.2. *Let $S \subset \mathbb{H}^1$ be a minimal surface with empty characteristic locus, then*

$$\mathcal{V}_{II}^H(S; \mathcal{X}) = \int_S \left\{ |\nabla^{H,S} F_{\mathcal{X}}|^2 + (2\mathcal{A} - \bar{w}^2) F_{\mathcal{X}}^2 \right\} d\sigma_H.$$

As a consequence, S is stable if and only if for any \mathcal{X} one has

$$\int_S (\bar{w}^2 - 2\mathcal{A}) F_{\mathcal{X}}^2 d\sigma_H \leq \int_S |\nabla^{H,S} F_{\mathcal{X}}|^2 d\sigma_H.$$

The following result is Corollary 15.4 in [11]. Let $\phi : \Omega \subset \mathbb{R}_{(u,v)}^2 \rightarrow \mathbb{R}$ give an intrinsic X_1 -graph S , we recall the formula (7) for the horizontal perimeter of S .

Corollary 3.3. *Let S be a C^2 minimal, intrinsic X_1 -graph, then for any \mathcal{X} one has*

$$\mathcal{V}_{II}^H(S; \mathcal{X}) = \int_{\Omega} \frac{\mathcal{B}_{\phi}(F_{\mathcal{X}})^2}{\sqrt{1 + \mathcal{B}_{\phi}(\phi)^2}} dudv - \int_{\Omega} \frac{\phi_v^2 + 2\mathcal{B}_{\phi}(\phi_v)}{\sqrt{1 + \mathcal{B}_{\phi}(\phi)^2}} F_{\mathcal{X}}^2 dudv,$$

where $F_{\mathcal{X}}$ is as in (17).

We next derive the second variation formula for special deformations of the intrinsic graph S . We consider compactly supported vector fields on S of the type $\mathcal{X} = \psi X_1$, where $\psi \in C_0^2(S)$. For this family of deformations we obtain from Corollary 3.3.

Theorem 3.4. *Let S be a C^2 minimal, intrinsic X_1 -graph, given by a function $\phi : \Omega \subset \mathbb{R}_{(u,v)}^2 \rightarrow \mathbb{R}$, then for any $\psi \in C_0^2(S)$ one has*

$$(19) \quad \begin{aligned} \mathcal{V}_{II}^H(S, \psi X_1) &= \int_{\Omega} \frac{\mathcal{B}_{\phi}(\psi)^2}{(1 + \mathcal{B}_{\phi}(\phi)^2)^{3/2}} dudv \\ &\quad - \int_{\Omega} \frac{\psi^2}{(1 + \mathcal{B}_{\phi}(\phi)^2)^{3/2}} \left(2(\mathcal{B}_{\phi}(\phi))_v - \phi_v^2 \right) dudv. \end{aligned}$$

Remark 3.5. *In the statement of the above result the function $\psi \in C_0^2(S)$. Slightly abusing the notation in the integral in the right-hand side of (19) we have continued to indicate with ψ the function in $C_0^2(\Omega)$ obtained by composing the original ψ with the parametrization of the surface S*

$$\Omega \ni (u, v) \longmapsto \left(\phi(u, v), u, v - \frac{u}{2}\phi(u, v) \right).$$

Proof. We note that with $\mathcal{X} = \psi X_1$, we have $a = \psi$, $b = k = 0$. We also recall, see [11], that for an intrinsic X_1 -graph one has

$$\bar{p} = \frac{1}{\sqrt{1 + \mathcal{B}_{\phi}(\phi)^2}}, \quad \bar{q} = -\frac{\mathcal{B}_{\phi}(\phi)}{\sqrt{1 + \mathcal{B}_{\phi}(\phi)^2}},$$

and therefore from (17) one has

$$(20) \quad F_{\mathcal{X}} = \frac{\psi}{\sqrt{1 + \mathcal{B}_{\phi}(\phi)^2}}.$$

From this formula a simple computation gives

$$\mathcal{B}_{\phi}(F_{\mathcal{X}}) = \frac{\mathcal{B}_{\phi}(\psi)}{\sqrt{1 + \mathcal{B}_{\phi}(\phi)^2}} - \frac{\mathcal{B}_{\phi}(\phi)\mathcal{B}_{\phi}(\mathcal{B}_{\phi}(\phi))}{(1 + \mathcal{B}_{\phi}(\phi)^2)^{3/2}}.$$

We now recall that the minimality of S is equivalent to ϕ being a solution of the double Burger equation

$$\mathcal{B}_{\phi}(\mathcal{B}_{\phi}(\phi)) = 0.$$

We thus conclude that

$$(21) \quad \mathcal{B}_{\phi}(F_{\mathcal{X}}) = \frac{\mathcal{B}_{\phi}(\psi)}{\sqrt{1 + \mathcal{B}_{\phi}(\phi)^2}}.$$

Using (20) and the identity

$$(\mathcal{B}_{\phi}(\phi))_v - \mathcal{B}_{\phi}(\phi_v) = \phi_v^2,$$

we thus obtain

$$- \int_{\Omega} \frac{\phi_v^2 + 2\mathcal{B}_{\phi}(\phi_v)}{\sqrt{1 + \mathcal{B}_{\phi}(\phi)^2}} F_{\mathcal{X}}^2 dudv = - \int_{\Omega} \frac{\psi^2}{(1 + \mathcal{B}_{\phi}(\phi)^2)^{3/2}} \left(2(\mathcal{B}_{\phi}(\phi))_v - \phi_v^2 \right) dudv.$$

On the other hand, (21) gives

$$\int_{\Omega} \frac{\mathcal{B}_{\phi}(F_{\mathcal{X}})^2}{\sqrt{1 + \mathcal{B}_{\phi}(\phi)^2}} dudv = \int_{\Omega} \frac{\mathcal{B}_{\phi}(\psi)^2}{(1 + \mathcal{B}_{\phi}(\phi)^2)^{3/2}} dudv.$$

Combining the last two equations we reach the desired conclusion. \square

Next, we apply Theorem 3.4 to the case of a strict intrinsic graphical strip as in Definition 1.9. We recall the diffeomorphism $\Psi : \mathbb{R} \times J \rightarrow \Omega = \Psi(\mathbb{R} \times J) \subset \mathbb{R}_{u,v}^2$ given by $\Psi(u, s) = (u, v) = (u, \frac{u^2}{2}G(s) + F(s)u + \sigma(s))$, see (10). As before, in the statement of the next result given a function $\psi \in C_0^2(S)$ slightly abusing the notation we will write $\psi \in C_0^2(\Omega)$. What we mean by this is the composition of the original ψ with the parametrization of the surface S

$$\Omega \ni (u, v) \longmapsto \left(\phi(u, v), u, v - \frac{u}{2}\phi(u, v) \right)$$

provided in Definition 1.9.

Corollary 3.6. *Let S be a strict intrinsic graphical strip defined by functions $F, G, \sigma \in C^2(J)$ and $\phi(u, v) = F(s(u, v)) + uG(s(u, v))$, as in Definition 1.9. One has for any $\psi \in C_0^2(S)$,*

$$(22) \quad \mathcal{V}_{II}^H(S, \psi X_1) = \int_{\mathbb{R} \times J} \left(\left(\frac{\partial}{\partial u}(\psi \circ \Psi)(u, s) \right)^2 \frac{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)}{(1 + G(s)^2)^{\frac{3}{2}}} + \frac{(\psi \circ \Psi)(u, s)^2}{(1 + G(s)^2)^{\frac{3}{2}}} \frac{F'(s)^2 - 2\sigma'(s)G'(s)}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)} \right) duds,$$

where we have indicated with $\Psi : \mathbb{R} \times J \rightarrow \Omega$ the diffeomorphism defined by (10).

Proof. We note that the proof of this theorem is similar to that of equation (5.12) of [3]. Since every strict intrinsic graphical strip is an intrinsic X_1 -graph, we can apply the second variation formula (19) in Theorem 3.4. In this formula we want to use the global diffeomorphism $\Psi : \mathbb{R} \times J \rightarrow \Omega$ to convert the integral on Ω to an integral on $\mathbb{R} \times J$. By (11)

$$\begin{aligned} \det J_{\Psi}(u, s) &= \det \begin{pmatrix} 1 & 0 \\ v_u & v_s \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ G(s)u + F(s) & G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s) \end{pmatrix} \\ &= G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s). \end{aligned}$$

We emphasize that since we are assuming that S is a strict graphical strip, then (9) is in force, and therefore the Jacobian of Ψ is always different from zero. Recall that we are also assuming that Ψ is globally one-to-one. The Inverse Function Theorem gives at every point $(u, v) = \Psi(u, s)$

$$J_{\Psi^{-1}}(u, v) = \begin{pmatrix} 1 & 0 \\ -\frac{G(s)u + F(s)}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)} & \frac{1}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)} \end{pmatrix}.$$

We thus have

$$(23) \quad s_u = -\frac{G(s)u + F(s)}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)}, \quad s_v = \frac{1}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)}.$$

Using (23) and the assumption that $\phi(u, v) = F(s) + uG(s)$, we thus find

$$\begin{aligned} \mathcal{B}_\phi(\phi) &= \phi_u + \phi\phi_v = G(s) + (G'(s)u + F'(s))s_u + \phi(G'(s)u + F'(s))s_v \\ &= G(s) - \frac{(F'(s) + uG'(s))(F(s) + uG(s))}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)} + \frac{(F'(s) + uG'(s))(F(s) + uG(s))}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)} \\ &= G(s). \end{aligned}$$

This gives,

$$\begin{aligned} (\mathcal{B}_\phi(\phi))_v &= G'(s)s_v = \frac{G'(s)}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)}, \\ (\phi_v)^2 &= (F'(s) + uG'(s))^2 s_v^2 = \left(\frac{F'(s) + uG'(s)}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)} \right)^2. \end{aligned}$$

Combining these formulas yields

$$2(\mathcal{B}_\phi(\phi))_v - \phi_v^2 = \frac{2\sigma'(s)G'(s) - F'(s)^2}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)}.$$

Substituting this into the second integral in the right-hand side of (19) gives

$$\mathcal{V}_{II}^H(S, \psi X_1) = \int_{\Omega} \frac{1}{(1 + G(s)^2)^{\frac{3}{2}}} \left((\mathcal{B}_\phi(\psi))^2 + \psi^2 \left(\frac{F'(s)^2 - 2\sigma'(s)G'(s)}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)} \right) \right) du dv.$$

Now, to complete the proof, we make the change of variable $(u, v) = \Psi(u, s)$, with $(u, s) \in \mathbb{R} \times J$. The Jacobian of such diffeomorphism is given by (11) which gives

$$dudv = \left(G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s) \right) duds.$$

Observe furthermore that

$$\mathcal{B}_\phi(\psi) = \psi_u + \phi\psi_v = \psi_u + (F + Gu)\psi_v = \psi_u + v_u\psi_v = \frac{\partial}{\partial u}\psi(u, v(u, s)) = \frac{\partial}{\partial u}(\psi \circ \Psi)(u, s).$$

Thus, we conclude that

$$\begin{aligned} \mathcal{V}_{II}^H(S, \psi X_1) &= \int_{\mathbb{R} \times J} \left(\left(\frac{\partial}{\partial u}(\psi \circ \Psi)(u, s) \right)^2 \frac{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)}{(1 + G(s)^2)^{\frac{3}{2}}} \right. \\ &\quad \left. + \frac{(\psi \circ \Psi)(u, s)^2}{(1 + G(s)^2)^{\frac{3}{2}}} \frac{F'(s)^2 - 2\sigma'(s)G'(s)}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)} \right) duds, \end{aligned}$$

which proves (22). □

4. PROOF OF THEOREM A: STRICT INTRINSIC GRAPHICAL STRIPS ARE UNSTABLE

In this section using the techniques of [10] and the modifications of [12], we construct a variation which strictly decreases the horizontal area of a strict intrinsic graphical strip (that is, we find a test function ψ for which $\mathcal{V}_{II}^H(S, \psi X_1) < 0$). To construct such a ψ we start by constructing a sequence ψ_k . We will show that for large enough k , we have $\mathcal{V}_{II}^H(S, \psi_k X_1) < 0$. This proves that such surfaces are unstable, thus establishing Theorem A.

For any given $\delta > 0$, we fix a function $\chi \in C_0^\infty(\mathbb{R})$ so that $0 \leq \chi(s) \leq 1$, $\chi(s) = 1$ for $|s| \leq \delta$, $\chi(s) = 0$ for $|s| \geq 2\delta$, and $|\chi'| \leq C = C(\delta)$. For each $k \in \mathbb{N}$, we let $\chi_k(s) = \chi(s/k)$ and hence

- $\chi_k(s) = 0$ for $|s| \geq 2\delta k$
- $\chi_k(s) = 1$ for $|s| \leq \delta k$
- $|\chi'_k(s)| \leq C/k$

Next, fix a function $\zeta \in C_0^\infty(\mathbb{R})$ with $\zeta \geq 0$, $\text{supp}(\zeta) = [-1, 1]$ and $\int_{\mathbb{R}} \zeta \, ds = 1$. Letting, $\zeta_k(s) = k\zeta(ks)$, we have that $\text{supp}(\zeta_k) = [-1/k, 1/k]$ and $\int_{\mathbb{R}} \zeta_k(s) \, ds = 1$. Let F, G and σ be the functions in Definition 1.9 with

$$(24) \quad F'(s)^2 - 2\sigma'(s)G'(s) < 0 \quad s \in J.$$

As we have mentioned in the introduction, without loss of generality we assume that $G', \sigma' > 0$ in J . We define $F_k = F \star \zeta_k$, $G_k = G \star \zeta_k$, $\sigma_k = \sigma \star \zeta_k$. Since F, G and σ are continuous on J . Shrinking J slightly if necessary, we may assume that they are uniformly continuous on \bar{J} . Therefore $F_k \rightarrow F$, $F'_k \rightarrow F'$, $G_k \rightarrow G$, $G'_k \rightarrow G'$, $\sigma_k \rightarrow \sigma$ and $\sigma'_k \rightarrow \sigma'$ uniformly on \bar{J} . The condition (24) now carries over to F_k, G_k, σ_k , that is, there is a positive integer k_o such that if $k > k_o$ (relabeling the sequence if necessary, we take $k_o = 1$) then for every $s \in J$, $F'_k(s)^2 - 2\sigma'_k(s)G'_k(s) < 0$. The left hand side of this inequality is precisely the discriminant of the quadratic expression in the variable u :

$$G'_k(s) \frac{u^2}{2} + F'_k(s)u + \sigma'_k(s).$$

Since the discriminant is strictly negative, $G'_k(s) \frac{u^2}{2} + F'_k(s)u + \sigma'_k(s)$ never vanishes for $u \in \mathbb{R}$ and $s \in J$. Next, we construct a sequence of test functions ψ_k to be used in the formula (22). We let

$$(25) \quad \psi_k(u, s) \stackrel{\text{def}}{=} \frac{\chi(s)\chi_k(u)}{\left(G'_k(s) \frac{u^2}{2} + F'_k(s)u + \sigma'_k(s)\right)^{\frac{1}{2}}},$$

We note that $\psi_k \in C_0^\infty(\mathbb{R} \times J)$ due to the above considerations. With ψ_k in hand, we analyze $\mathcal{V}_{II}^H(S, \psi_k X_1)$. Before proceeding to the computations, we remark that the function ψ in (22) is defined on $\Omega = \Psi(\mathbb{R} \times J)$. Our ψ_k 's have been already defined on the (u, s) space, that is on $\mathbb{R} \times J$. Therefore, occurrences of $\psi \circ \Psi$ in (22) will be replaced by ψ_k in the proof of the subsequent two lemmas. We start with the second term in the right hand side of (22).

Lemma 4.1. *We have*

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\mathbb{R} \times J} \frac{\psi_k(u, s)^2}{(1 + G(s)^2)^{\frac{3}{2}}} \frac{F'(s)^2 - 2\sigma'(s)G'(s)}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)} du ds \\ = -2\pi \int_J \frac{\chi(s)^2}{(1 + G(s)^2)^{\frac{3}{2}}} \frac{G'(s)}{(2\sigma'(s)G'(s) - F'(s)^2)^{\frac{1}{2}}} ds \end{aligned}$$

Proof. Substituting the quantity $\psi \circ \Psi$ with ψ_k in the second term of the right hand side of (22) and recalling the definition of ψ_k we have

$$\begin{aligned} (26) \quad \lim_{k \rightarrow \infty} \int_{\mathbb{R} \times J} \frac{\psi_k(u, s)^2}{(1 + G(s)^2)^{\frac{3}{2}}} \frac{F'(s)^2 - 2\sigma'(s)G'(s)}{G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)} du ds \\ = \lim_{k \rightarrow \infty} \int_J \chi(s)^2 \frac{F'(s)^2 - 2\sigma'(s)G'(s)}{(1 + G(s)^2)^{\frac{3}{2}}} \\ \times \left(\int_{\mathbb{R}} \frac{\chi_k(u)^2}{(G'_k(s)\frac{u^2}{2} + F'_k(s)u + \sigma'_k(s))(G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s))} du \right) ds \\ = \int_J \chi(s)^2 \frac{F'(s)^2 - 2\sigma'(s)G'(s)}{(1 + G(s)^2)^{\frac{3}{2}}} \left(\int_{\mathbb{R}} \frac{1}{(G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s))^2} du \right) ds. \end{aligned}$$

In the above, we have used the fact that since for each $u \in \mathbb{R}$,

$$G'_k(s)\frac{u^2}{2} + F'_k(s)u + \sigma'_k(s) \longrightarrow G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s) \quad \text{as } k \rightarrow \infty$$

uniformly for $s \in \bar{J}$, and the latter quantity never vanishes, we have

$$\frac{1}{2}|G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)| < |G'_k(s)\frac{u^2}{2} + F'_k(s)u + \sigma'_k(s)| < 2|G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)|.$$

Hence, Lebesgue dominated convergence theorem allows taking the limit inside the integral. Next, we want to compute the integral

$$\int_{\mathbb{R}} \frac{1}{(G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s))^2} du.$$

Using standard integration techniques we obtain

$$\int \frac{1}{(Au^2 + Bu + C)^2} du = \frac{2Au + B}{(4AC - B^2)(Au^2 + Bu + C)} + \frac{4A}{(4AC - B^2)^{\frac{3}{2}}} \arctan\left(\frac{2Au + B}{\sqrt{4AC - B^2}}\right).$$

This implies if $A > 0$

$$\int_{\mathbb{R}} \frac{1}{(Au^2 + Bu + C)^2} du = \frac{4\pi A}{(4AC - B^2)^{\frac{3}{2}}}.$$

Since we have that $G'(s) > 0$, letting $A = G'(s)/2$, $B = F'(s)$ and $C = \sigma'(s)$ we have

$$(27) \quad \int_{\mathbb{R}} \frac{1}{(G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s))^2} du = 2\pi \frac{G'(s)}{(2\sigma'(s)G'(s) - F'(s)^2)^{\frac{3}{2}}}.$$

Substituting (27) in (26) we reach the desired conclusion. \square

Now we turn to the first term in the right hand side of (22).

Lemma 4.2. *We have*

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\mathbb{R} \times J} \left(\left(\frac{\partial \psi(u, s)}{\partial u} \right)^2 \frac{G'(s) \frac{u^2}{2} + F'(s)u + \sigma'(s)}{(1 + G(s)^2)^{\frac{3}{2}}} \right) du ds \\ = \frac{\pi}{2} \int_J \frac{\chi(s)^2}{(1 + G(s)^2)^{\frac{3}{2}}} \frac{G'(s)}{(2\sigma'(s)G'(s) - F'(s)^2)^{\frac{1}{2}}} ds \end{aligned}$$

Proof. Again, we closely follow the development in [12]. By recalling (25) we first obtain

$$\frac{\partial \psi_k}{\partial u}(u, s) = \frac{\chi(s)}{2} \left(\frac{2\chi'_k(u)Q_k(u, s) - \chi_k(u)D_k(u, s)}{Q_k(u, s)^{\frac{3}{2}}} \right),$$

where we have let

$$Q_k(u, s) = G'_k(s) \frac{u^2}{2} + F'_k(s)u + \sigma'_k(s) \quad \text{and} \quad D_k(u, s) = uG'_k(s) + F'_k(s).$$

For the computations that follow, it is convenient to also let

$$Q(u, s) = G'(s) \frac{u^2}{2} + F'(s)u + \sigma'(s) \quad \text{and} \quad D(u, s) = \frac{\partial}{\partial u} Q(u, s) = uG'(s) + F'(s).$$

It follows that

$$\left(\frac{\partial \psi_k}{\partial u}(u, s) \right)^2 = \chi(s)^2 \left(\frac{\chi'_k(u)^2}{Q_k(u, s)} - \frac{1}{2} (\chi_k(u)^2)' \frac{D_k(u, s)}{Q_k(u, s)^2} + \frac{1}{4} \chi_k(u)^2 \frac{D_k(u, s)^2}{Q_k(u, s)^3} \right).$$

Substituting the quantity $\psi \circ \Psi$ in the first term of the right hand side of (22), and using the above expression for $\psi_{k,u}$, we have

$$\int_{\mathbb{R} \times J} \left(\frac{\partial \psi_k(u, s)}{\partial u} \right)^2 \frac{G'(s) \frac{u^2}{2} + F'(s)u + \sigma'(s)}{(1 + G(s)^2)^{\frac{3}{2}}} du ds = \int_J \frac{\chi(s)^2}{(1 + G'(s)^2)^{\frac{3}{2}}} \left(\boxed{1} + \boxed{2} + \boxed{3} \right) ds$$

where,

$$\begin{aligned} \boxed{1} &= \int_{\mathbb{R}} \chi'_k(u)^2 \frac{Q(u, s)}{Q_k(u, s)} du, & \boxed{2} &= -\frac{1}{2} \int_{\mathbb{R}} (\chi_k^2(u))' Q(u, s) \frac{D_k(u, s)}{Q_k(u, s)^2} du, \\ \boxed{3} &= \frac{1}{4} \int_{\mathbb{R}} \chi_k(u)^2 Q(u, s) \frac{D_k(u, s)^2}{Q_k(u, s)^3} du. \end{aligned}$$

Since $|\chi'_k(u)| \leq \frac{C}{k}$, by Lebesgue dominated convergence theorem we have

$$(28) \quad \lim_{k \rightarrow \infty} \boxed{1} = 0$$

In addition, since $D_k(u, s) \rightarrow D(u, s)$, $Q_k(u, s) \rightarrow Q(u, s)$, and $\chi_k(s) \rightarrow 1$ when $k \rightarrow \infty$, we obtain

$$\begin{aligned}
 (29) \quad \lim_{k \rightarrow \infty} \boxed{3} &= \frac{1}{4} \int_{\mathbb{R}} \frac{D(u, s)^2}{Q(u, s)^2} du = -\frac{1}{4} \int_{\mathbb{R}} \frac{\partial}{\partial u} Q(u, s) \frac{\partial}{\partial u} \left(\frac{1}{Q(u, s)} \right) du \\
 &= \frac{1}{4} \int_{\mathbb{R}} \frac{\partial^2 Q(u, s)}{\partial u^2} \frac{1}{Q(u, s)} du = \frac{1}{4} \int_{\mathbb{R}} \frac{G'(s)}{G'(s) \frac{u^2}{2} + F'(s)u + \sigma'(s)} du \\
 &= \frac{\pi G'(s)}{(2\sigma'(s)G'(s) - F'(s)^2)^{\frac{1}{2}}}.
 \end{aligned}$$

The third equality above is obtained by integration by parts whereas in the last equality, we have used the fact that $G'(s) > 0$ and standard calculus techniques. Now we turn to the quantity $\boxed{2}$.

$$(30) \quad \lim_{k \rightarrow \infty} \boxed{2} = -\lim_{k \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}} (\chi_k(u)^2)' Q(u, s) \frac{D_k(u, s)}{Q_k(u, s)} du$$

$$\begin{aligned}
 (31) \quad &= -\lim_{k \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}} \chi_k(u)^2 \frac{\partial}{\partial u} \left(\frac{Q(u, s) D_k(u, s)}{Q_k(u, s)^2} \right) du \\
 &= -\lim_{k \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}} \chi_k(u)^2 \left(\frac{Q_u(u, s) D_k(u, s)}{Q_k(u, s)^2} \right. \\
 &\quad \left. + \frac{Q(u, s) D_{k,u}(u, s)}{Q_k(u, s)^2} - 2 \frac{Q(u, s) D_k(u, s) Q_{k,u}(u, s)}{Q_k(u, s)^3} \right) du \\
 &= -\frac{1}{2} \int_{\mathbb{R}} \frac{Q_u(u, s) D(u, s)}{Q(u, s)^2} + \frac{D_u(u, s)}{Q(u, s)} - 2 \frac{D(u, s) Q_u(u, s)}{Q(u, s)^2} du \\
 &= -\frac{1}{2} \int_{\mathbb{R}} \frac{G'(s)}{Q(u, s)} du - \frac{1}{2} \int_{\mathbb{R}} \frac{\partial}{\partial u} Q(u, s) \frac{\partial}{\partial u} \left(\frac{1}{Q(u, s)} \right) du \\
 &= -\frac{1}{2} \int_{\mathbb{R}} \frac{G'(s)}{Q(u, s)} du + \frac{1}{2} \int_{\mathbb{R}} \frac{Q_{uu}(u, s)}{Q(u, s)} du = 0,
 \end{aligned}$$

since $Q_{uu}(u, s) = G'(s)$. Combining (28), (29) and (30), we obtain the desired conclusion. \square

Combining (22) with Lemmas 4.1 and 4.2 we can now prove Theorem A in the introduction.

Proof of Theorem A. Let ψ_k be the function constructed in (25) and consider $\psi_k \circ \Psi^{-1} \in C_0^2(\Omega)$, where Ψ is the diffeomorphism in (10). If we lift this function to the surface, and by abuse of notation we continue to indicate with ψ_k such lifted function, we obtain a function in $C_0^2(S)$. From Corollary 3.6, Lemmas 4.1, 4.2 and the fact that $G'(s) > 0$ on J we deduce that

$$\lim_{k \rightarrow \infty} \mathcal{V}_{II}^H(S, (\psi_k X_1)) = \left(\frac{\pi}{2} - 2\pi \right) \int_J \frac{\chi(s)^2}{(1 + G(s)^2)^{\frac{3}{2}}} \frac{G'(s)}{(2\sigma'(s)G'(s) - F'(s)^2)^{\frac{1}{2}}} ds < 0.$$

Therefore, for large enough k we have $\mathcal{V}_{II}^H(S, \psi_k X_1) < 0$. This completes the proof. \square

5. PROOF OF THEOREM B: EXISTENCE OF STRICT INTRINSIC GRAPHICAL STRIPS

The main objective of this section is to establish the crucial Theorem B in the introduction. The proof of this result will be accomplished in several steps. Before we turn to the general discussion it will be helpful for the understanding of Definition 1.9 to analyze directly the situation of the surfaces introduced in (12).

5.1. The sub-Riemannian catenoid is unstable. In what follows we illustrate the construction of a strict intrinsic graphical strip for the hyperboloids of revolution in \mathbb{H}^1 described by (12). This is an interesting example of a complete embedded minimal surface in \mathbb{H}^1 which has empty characteristic locus and which is neither a graph over any plane, nor an intrinsic graph in the sense of [17], [18]. Such surface should be considered as the sub-Riemannian analogue of the *catenoid* in the classical theory of minimal surfaces. We emphasize that (12) does not contain any strict graphical strip in the sense of [12], and therefore the results in that paper do not apply to it. Instead, as a consequence of the following calculations and Theorem A we are able to conclude that the surface (12) is unstable. To fix the ideas we will focus on the case $a = 0, b = 4$, in which case we have from (12)

$$(32) \quad t^2 - \frac{1}{4}((x^2 + y^2) - 1).$$

A local parametrization of S as a ruled surface is given by

$$(33) \quad \theta(r, s) = \left(r \sin s + \cos s, r \cos s - \sin s, \frac{r}{2} \right), \quad r \in \mathbb{R}, -\pi < s < \pi.$$

Clearly, if we consider the open set $U = \mathbb{R} \times (-\pi, \pi)$, then $\theta(U)$ does not cover the whole catenoid, but this fact is inconsequential for what follows. We now consider the projection mapping $\Pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \times \{0\}$ given by

$$\Pi(x, y, t) = (0, y, t + \frac{xy}{2}).$$

We thus have

$$\begin{aligned} \Pi(\theta(U)) &= \left(0, r \cos s - \sin s, \frac{r}{2} + \frac{(r \sin s + \cos s)(r \cos s - \sin s)}{2} \right) \\ &= \left(0, r \cos s - \sin s, \frac{r^2}{2} \sin s \cos s + r \cos^2 s - \frac{\sin s \cos s}{2} \right) \end{aligned}$$

We now define a mapping from the (r, s) to the (u, s) plane by setting

$$\Lambda(r, s) = (r \cos s - \sin s, s).$$

With $\epsilon \in (0, \pi/4)$ to be chosen later, and

$$U_\epsilon = \mathbb{R} \times (-\epsilon, \epsilon),$$

it is clear that Λ is a C^∞ diffeomorphism of U_ϵ onto its image $\Lambda(U_\epsilon)$. Notice that, thanks to the fact that $1 < \sec s < \sqrt{2}$ for $-\epsilon < s < \epsilon$, we have $\Lambda(U_\epsilon) = U_\epsilon$. Let us notice that the inverse diffeomorphism is given by

$$(r, s) = \Lambda^{-1}(u, s) = \left(\frac{u + \sin s}{\cos s}, s \right) = (u \sec s + \tan s, s).$$

Next, we define a mapping from the (r, s) to the (u, v) plane by setting

$$\Phi(r, s) = (u, v)$$

with

$$(34) \quad \begin{cases} u = r \cos s - \sin s, \\ v = \frac{r^2}{2} \sin s \cos s + r \cos^2 s - \frac{\sin s \cos s}{2}. \end{cases}$$

We want to show that Φ is a diffeomorphism onto its image. To see this we take the composition $\Psi = \Phi \circ \Lambda^{-1} : U_\epsilon \rightarrow \mathbb{R}^2$, which maps the (u, s) to the (u, v) plane. We obtain

$$(u, v) = \Psi(u, s) = \Phi(\Lambda^{-1}(u, s)) = \left(u, G(s) \frac{u^2}{2} + F(s)u + \sigma(s) \right),$$

where

$$\begin{cases} G(s) = \tan s, \\ F(s) = \sec s, \\ \sigma(s) = \frac{\tan s}{2}. \end{cases}$$

Let us observe that the determinant of the Jacobian of $\Psi(u, s)$ at any point $(u, s) \in U_\epsilon$ is given by

$$G'(s) \frac{u^2}{2} + F'(s)u + \sigma'(s) = \frac{\sec^2 s}{2} [u^2 + 2 \sin s u + 1].$$

Since for the quadratic expression within the square brackets we have

$$\Delta = \sin^2 s - 1 < 0,$$

it is clear that such determinant never vanishes. We next show that Ψ is globally one-to-one on U_ϵ provided that $\epsilon > 0$ is chosen sufficiently small. Suppose by contradiction that $(u, s), (u', s') \in U_\epsilon$, $(u, s) \neq (u', s')$, and $\Psi(u, s) = \Psi(u', s')$. It cannot be $u \neq u'$ (since then $\Psi(u, s) \neq \Psi(u', s')$). We can thus suppose that $s \neq s'$, but $u = u'$. Since $\tan s$ is strictly increasing, $s \neq s'$ implies $G(s) \neq G(s')$. But then we must have

$$(35) \quad u^2 + 2 \frac{F(s) - F(s')}{G(s) - G(s')} u + 1 = 0.$$

We would like to show that there exists $0 < \epsilon < \pi/4$ such that for every $s, s' \in (-\epsilon, \epsilon)$, with $s \neq s'$, one has

$$(36) \quad \left(\frac{F(s) - F(s')}{G(s) - G(s')} \right)^2 < 1.$$

If this were the case then we would reach a contradiction since this implies that the equation (35) has no real solutions. Now (36) is equivalent to

$$(37) \quad \left(\frac{\sec s - \sec s'}{\tan s - \tan s'} \right)^2 < 1,$$

for every $s, s' \in (-\epsilon, \epsilon)$, with $s \neq s'$. Without restriction we can assume $s < s'$, otherwise we reverse their role. Using the mean value theorem we find that for some $\xi, \xi' \in (s, s') \subset (-\epsilon, \epsilon)$

$$\frac{\sec s - \sec s'}{\tan s - \tan s'} = \frac{\sec \xi \tan \xi}{1 + \tan^2 \xi'} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0^+.$$

Therefore, we can achieve (36) provided that $\epsilon > 0$ is sufficiently small. Having fixed ϵ in such a way, the map $\Psi : U_\epsilon \rightarrow \mathbb{R}^2$ defines a C^∞ diffeomorphism from the (u, s) plane onto its image

$V_\epsilon \stackrel{def}{=} \Psi(U_\epsilon)$, which is an open set of the (u, v) plane. We now claim that there exists $\delta = \delta(\epsilon) > 0$ such that

$$(38) \quad \Omega \stackrel{def}{=} \mathbb{R} \times (-\delta, \delta) \subset V_\epsilon.$$

To prove (38) it suffices to show that, as s ranges over the interval $(-\epsilon, \epsilon)$ the v -coordinate of the vertices of the parabolas $v = v(u, s) = \frac{\tan s}{2}u^2 + \sec s u + \frac{\tan s}{2}$ are uniformly bounded away from zero. Let us notice that the line $s = 0$ in the (u, s) plane is mapped to the line $v = u$ of the (u, v) plane. For $s \neq 0$ the v coordinate of the vertex of the parabola is given by

$$v(s) = -\frac{\sec^2 s(1 - \sin^2 s)}{2 \tan s} = -\frac{\cot s}{2}.$$

Now on the interval $0 < s < \epsilon$ we have $v(s) \rightarrow -\infty$ as $s \rightarrow 0^+$, whereas on $(-\epsilon, 0)$ we have $v(s) \rightarrow +\infty$ as $s \rightarrow 0^-$. Since $\cot s$ is strictly decreasing on $(-\epsilon, \epsilon)$, we conclude that if we take

$$\delta = \delta(\epsilon) = \frac{\cot \epsilon}{2},$$

then (38) is verified. Since the composition of diffeomorphisms is a diffeomorphism as well, we conclude that

$$\Phi \stackrel{def}{=} \Psi \circ \Lambda : U_\epsilon \rightarrow \Omega \subset \mathbb{R}_{u,v}^2$$

is also a diffeomorphism. At this point, using the inverse diffeomorphism $\Phi^{-1} : \Omega \rightarrow U_\epsilon$, we define

$$\phi(u, v) = \theta_1(\Phi^{-1}(u, v)), \quad (u, v) \in \Omega,$$

where $\theta_1(r, s) = r \sin s + \cos s$ is the first component of the map θ in (33). Notice that

$$\phi(u, v) = F(s(u, v)) + G(s(u, v))u,$$

where $(r(u, v), s(u, v))$ is the inverse diffeomorphism of (34).

With this definition of ϕ we now see that portion of the catenoid which is parametrized by θ on the open set $U_\epsilon = \mathbb{R} \times (-\epsilon, \epsilon)$ is in fact given as the X_1 -graph

$$\left(\phi(u, v), u, v - \frac{u}{2}\phi(u, v) \right),$$

for $(u, v) \in \Omega$. Finally, let us notice that such piece of the surface is a strict intrinsic graphical strip in the sense of Definition 1.9 since the condition

$$F'(s)^2 < 2G'(s)\sigma'(s), \quad s \in (-\epsilon, \epsilon),$$

is verified.

5.2. Proof of Theorem B. The above analysis should allow the reader a clear understanding of the motivation behind the Definition 1.9 of strict intrinsic graphical strip. Our next objective is proving that, similarly to the sub-Riemannian catenoid, every complete minimal surface without boundary and with empty characteristic locus contains a strict intrinsic graphical strip, unless the surface is a vertical plane. In this general case the construction of the strict graphical strip is more difficult. Our approach hinges on the following basic representation theorem for minimal surfaces which is a consequence of the results in [20], and which has already proved crucial in [12].

Theorem 5.1. *Let S be a C^2 complete embedded non-characteristic minimal surface without boundary and assume that it is not a vertical plane. Let $g_0 \in S$ be a point admitting a neighborhood (in S) that may be written as a graph over the plane $t = 0$. There exist a neighborhood U of g_0 , an interval J , and functions $h_0 \in C^2(J)$, $\gamma \in C^3(J, \mathbb{R}^2)$, with $|\gamma'(s)| = 1$ for $s \in J$, such that U is parameterized by $\mathcal{L} : \mathbb{R} \times J \rightarrow \mathbb{H}$*

$$(39) \quad \mathcal{L}(r, s) = \left(\gamma(s) + r(\gamma')^\perp(s), h_0(s) - \frac{r}{2} \gamma(s) \cdot \gamma'(s) \right)$$

for $s \in J, r \in \mathbb{R}$. Moreover, with $W_0(s) = h'_0(s) + \frac{1}{2} \gamma' \cdot \gamma^\perp(s)$ and $\kappa(s) = \gamma'' \cdot (\gamma')^\perp$, we have that

$$(40) \quad 1 - 2W_0(s)\kappa(s) < 0, \quad s \in J.$$

The proof of Theorem 5.1 will be presented after Corollary 5.5 below. We first develop some preparatory results.

Lemma 5.2. *Let $D \subset \mathbb{R}^2$ be an open set, $g \in C^2(D)$, and consider the C^2 map $G : D \rightarrow \mathbb{H}^1$ given by $G(x, y) = (x, y, g(x, y))$. Suppose that $S = G(D)$ is a non-characteristic minimal surface. Then S is foliated by horizontal straight lines which are the integral curves of $\nu_H^\perp = \bar{q}X_1 - \bar{p}X_2$.*

Proof. Writing S as the level set $\phi(x, y, t) = g(x, y) - t = 0$ we have that

$$\nu_H = \bar{p} X_1 + \bar{q} X_2,$$

where

$$\bar{p} = \frac{X_1 \phi}{\sqrt{(X_1 \phi)^2 + (X_2 \phi)^2}}, \quad \bar{q} = \frac{X_2 \phi}{\sqrt{(X_1 \phi)^2 + (X_2 \phi)^2}}.$$

The reader should keep in mind here that

$$(41) \quad p = X_1 \phi = X_1 g + \frac{y}{2} = g_x + \frac{y}{2}, \quad q = X_2 \phi = X_2 g - \frac{x}{2} = g_y - \frac{x}{2}.$$

We emphasize that the assumption that S be non-characteristic is equivalent to

$$W = \sqrt{(X_1 \phi)^2 + (X_2 \phi)^2} \neq 0 \quad \text{on } D.$$

By Proposition 2.3 we see that assumption that S be minimal reads

$$X_1 \bar{p} + X_2 \bar{q} = 0,$$

which, using the fact that \bar{p}, \bar{q} are independent of t , is equivalent to

$$(42) \quad \text{div} V = \bar{p}_x + \bar{q}_y = 0.$$

Here, we view $V = \bar{p} \partial_x + \bar{q} \partial_y$ as a vector field on D and div is the Euclidean divergence. We now claim that if $c(s) = (c_1(s), c_2(s)) \subset D$ is an integral curve in D of $V^\perp = \bar{q} \partial_x - \bar{p} \partial_y$, then $C(s) = (c_1(s), c_2(s), g(c_1(s), c_2(s)))$ must be an integral curve of ν_H^\perp on S . To see this suppose that $c'(s) = V^\perp(c(s))$, which means $c'_1(s) = \bar{q}(c(s))$, $c'_2(s) = -\bar{p}(c(s))$. Now from these equations and from (41) one has

$$(43) \quad c'_1 \left(g_x(c) + \frac{c_2}{2} \right) + c'_2 \left(g_y(c) - \frac{c_1}{2} \right) = \bar{q}(c)p(c) - \bar{p}(c)q(c) = \frac{q(c)p(c) - p(c)q(c)}{W} = 0,$$

where for simplicity we have omitted the variable s when writing c instead of $c(s)$. Now,

$$C'(s) = (c'_1(s), c'_2(s), g_x(c_1(s), c_2(s))c'_1(s) + g_y(c_1(s), c_2(s))c'_2(s)).$$

Using the formula

$$(44) \quad aX_1 + bX_2 + cT = \left(a, b, c + \frac{bx - ay}{2} \right),$$

which allows to pass from the standard representation in terms of the Cartesian coordinates in \mathbb{H}^1 to that with respect to the orthonormal basis $\{X_1, X_2, T\}$, we find

$$(45) \quad \begin{aligned} C'(s) &= c'_1(s)X_1(c(s)) + c'_2(s)X_2(c(s)) \\ &+ \left(\nabla g(c_1(s), c_2(s)) \cdot (c'_1(s), c'_2(s)) + \frac{c'_1(s)c_2(s) - c_1(s)c'_2(s)}{2} \right) T. \end{aligned}$$

From (43) we conclude that the component of $C'(s)$ with respect to T is identically equal to zero, and therefore

$$C'(s) = c'_1(s)X_1(c(s)) + c'_2(s)X_2(c(s)) = \bar{q}(c(s))X_1(c(s)) - \bar{p}(c(s))X_2(c(s)) = \nu_H^\perp(c(s)).$$

This proves the claim.

Since V is a unit vector field, we have that

$$(46) \quad \bar{p} \bar{p}_x + \bar{q} \bar{q}_x = \bar{p} \bar{p}_y + \bar{q} \bar{q}_y = 0.$$

Combining equations (46) and (42), we conclude that if $c(s) = (c_1(s), c_2(s))$ is an integral curve of V^\perp , with $c(0) = z = (x, y) \in D$, then

$$\frac{d}{ds} V^\perp(c(s)) \equiv 0,$$

and therefore $V^\perp(c(s)) \equiv V^\perp(c(0)) = V^\perp(z)$. It follows that

$$c(t) = z + sV^\perp(z),$$

i.e., $c(s)$ is a segment of straight line in D^2 passing through z . If we write $c(t) = (x + as, y + bs)$, then

$$C(s) = (x + as, y + bs, g(x + as, y + bs)),$$

and so

$$C'(s) = (a, b, ag_x + bg_y).$$

At this point we note that the vanishing of the T component in (43) now implies that

$$\frac{d}{ds} g(x + as, y + bs) = g_x a + g_y b = -\frac{c'_1(s)c_2(s) - c_1(s)c'_2(s)}{2} = \frac{bx - ay}{2}.$$

This gives

$$g(x + as, y + bs) = g(z) + \frac{bx - ay}{2}s,$$

and therefore,

$$C(s) = \left(x + as, y + bs, g(z) + \frac{bx - ay}{2}s \right),$$

i.e., $C(s)$ is a straight line segment in \mathbb{H}^1 . □

Lemma 5.3. *Suppose S be a C^2 non-characteristic minimal surface such that no open subset of S may be written as a graph over the xy -plane. Then, S is a piece of a vertical plane and, hence, is foliated by horizontal straight lines which are the integral curves of ν_H^\perp .*

Proof. Let $(x_0, y_0, t_0) \in S$ and let $U \subset \mathbb{H}^1$ be an open neighborhood of (x_0, y_0, t_0) such that $S \cap U = \{(x, y, t) \in U \mid \phi(x, y, t) = 0\}$ for a $\phi \in C^2(U)$ having $\nabla \phi \neq 0$ in U . By the assumption that no open subset of S may be written as a graph over the xy -plane, we see that it must be $\phi_t = 0$ in U . Then, $\phi(x, y, t) = \phi_0(x, y)$ in U and therefore $S \cap U$ is a portion of a ruled surface over a curve c in the xy -plane. Furthermore, due to the special structure of ϕ one easily recognizes that the assumption that S be H -minimal now translates into the fact that ϕ_0 must satisfy the classical minimal surface equation

$$\operatorname{div} \left(\frac{\nabla \phi_0}{\sqrt{1 + |\nabla \phi_0|^2}} \right) = 0 ,$$

on the open set $\tilde{U} = \pi(U) \subset \mathbb{R}^2$, where $\pi(x, y, t) = (x, y)$. This equation is in fact equivalent to

$$(47) \quad (1 + \phi_{0,y}^2)\phi_{0,xx} - 2\phi_{0,x}\phi_{0,y}\phi_{0,xy} + (1 + \phi_{0,x}^2)\phi_{0,yy} = 0 .$$

Since $\nabla \phi_0 \neq 0$ in U , by the Implicit Function Theorem, we may locally describe the curve c by either $y = g(x)$ or $x = f(y)$. In the former case, we have $\phi_0(x, y) = y - g(x)$, and thus (47) implies that $g'' = 0$. We conclude that there exists an open set $V \subset \mathbb{H}^1$ containing (x_0, y_0, t_0) such that $S \cap V$ is a piece of a vertical plane. The second case leads to the same conclusion. By the assumption that S be C^2 we now conclude that if for two such different open sets V_1, V_2 one has $V_1 \cap V_2 \neq \emptyset$, then the two corresponding portions of planes $S \cap V_1$ and $S \cap V_2$ must be part of the same plane. This completes the proof. \square

In the next lemma we combine into a single result the two different situations considered in Lemmas 5.2 and 5.3.

Lemma 5.4. *Let S be a C^2 minimal surface in \mathbb{H}^1 with empty characteristic locus, and let p be a point in the interior of S (in the relative topology). Then, there exists a neighborhood Δ of p in S which is foliated by horizontal straight line segments which are integral curves of ν_H^\perp .*

Proof. For every $p \in \overset{\circ}{S}$, there exists an open set $U \subset \mathbb{H}^1$ and a $\phi \in C^2(U)$ such that $\nabla \phi \neq 0$ in U and $\Sigma = S \cap U = \{(x, y, t) \in U \mid \phi(x, y, t) = 0\}$. Let $S_1 = \{(x, y, t) \in \Sigma \mid \phi_t(x, y, t) \neq 0\}$, $S_2 = \{(x, y, t) \in \Sigma \mid \phi_t(x, y, t) = 0\}$. Notice that, either $\phi_t \equiv 0$ on Σ and in such case $S_2 = \Sigma$ is a vertical cylinder over a curve in the xy plane, or there exists an open set $V \subset \mathbb{H}^1$ such that $S_2 \cap V$ is a C^1 curve in \mathbb{H}^1 . In the former case we can invoke Lemma 5.3 to conclude that $\Delta = \Sigma$ is foliated by horizontal straight line segments which are integral curves of ν_H^\perp . We are thus left with the case in which $S_1 \neq \emptyset$. By shrinking Σ if necessary we can assume that $\Sigma = S_1 \cup S_2$, where S_2 is a C^1 curve.

In our arguments, we consider integral curves of ν_H^\perp passing through points on the surface S . To make this notion precise, we recall that as S is a C^2 submanifold of $\mathbb{H}^1 = \mathbb{R}^3$, every point $p \in S$ is contained in a coordinate chart $i : D \subset \mathbb{R}^2 \rightarrow S$ with $i \in C^2(D)$. For any C^1 vector field, U_0 , defined on $i(D)$, the integral curve of U_0 passing through $q \in i(D)$ is simply $i(\gamma)$ where $\gamma \subset D$ is a solution to the initial value problem:

$$\begin{aligned} \gamma'(t) &= i_*^{-1}(U_0)(\gamma(t)) \\ \gamma(0) &= i^{-1}(q). \end{aligned}$$

Direct calculation then shows that

$$\frac{d}{dt}i(\gamma) = i_*i_*^{-1}U_0(\gamma(t)) = U_0(i(\gamma(t))),$$

and $i(\gamma(0)) = i(i^{-1}(q)) = q$. As U_0 (and hence i^*U_0) is C^1 , the standard theorems concerning solutions to ODE apply to the integral curves of U_0 on S . In particular, we may conclude that given $q \in S$, there exists (at least for a short time) a unique integral curve of U_0 . Similarly, we conclude that integral curves of U_0 on S have continuous dependence on parameters.

By Lemma 5.2, each point in S_1 is contained in a neighborhood which is foliated by straight line segments which are integral curves of ν_H^\perp . Thus, those portions of integral curves of ν_H^\perp contained in S_1 are at least piecewise linear. By the fact that ν_H^\perp is C^1 and the uniqueness of solutions to ode's, we must have that these portions of integral curves are straight lines. We may extend each such line segment maximally within S_1 . If a limit point of a maximally extended line segment were in S_1 , we could apply Lemma 5.2 to extend it further, violating the assumption that we had extended maximally. Thus we conclude that the limit points of the line segment are in $\partial S_1 \cup S_2$.

Consider $p \in S_2$ and let c be the integral curve of ν_H^\perp with $c(0) = p$. Let B_ϵ be the metric ball of radius ϵ centered at p and $c_\epsilon = c \cap B_\epsilon$. Then, there exists an $\epsilon > 0$ sufficiently small so that one of the following possibilities occurs:

- (1) $c_\epsilon \cap S_2$ is closed and has no interior;
- (2) $c_\epsilon \cap S_2$ is closed with nonempty interior and p is in the interior;
- (3) $c_\epsilon \cap S_2$ is closed with nonempty interior and p is contained in the boundary of the interior of $c_\epsilon \cap S_2$.

In the first case, $c_\epsilon \cap S_1$ is open and dense in c_ϵ . By Lemma 5.2, every point in $c_\epsilon \cap S_1$ is contained in an open line segment which is a subset of c_ϵ . As $c_\epsilon \cap S_2$ is closed and is contained in the boundary of $c_\epsilon \cap S_1$, we conclude that c_ϵ is piecewise linear. By the smoothness of ν_H^\perp and the uniqueness of solutions to ODE, we conclude c_ϵ is a single straight line segment.

In the second case, we may shrink ϵ so that $c_\epsilon \cap S_2 = c_\epsilon$ and S_2 divides $B_\epsilon \cap S$ into exactly two pieces N_1, N_2 . We next show that if $q \in N_1$ is contained in a line segment, $L \subset N_1$, which reaches the boundary of N_1 then the length of L is at least $2(\epsilon - \delta)$ where δ is the Euclidean distance from p to q . Observe that the endpoints of L can not be in S_2 . If one were in S_2 , then by the uniqueness of solutions of ODE, we conclude that L and S_2 coincide. This contradicts our assumption that $q \notin S_2$. Thus, L must be a line segment in B_ϵ which has both its boundary points in ∂B_ϵ . By construction, the Euclidean distance from p to the endpoints of L is ϵ . Denoting the Euclidean distance from p to q by δ , the triangle inequality implies that the length of L is at least $2(\epsilon - \delta)$.

Let $q_i \in N_1$ be a sequence of points converging to p and let L_i be the maximal line segment which is the integral curve of ν_H^\perp through q_i which is contained in N_1 . By the continuous dependence on parameters of the solutions to an ODE and the fact the ν_H^\perp is C^1 , we know $L = \lim_{i \rightarrow \infty} L_i$ exists and is an integral curve of ν_H^\perp passing through p . Moreover, since L is the limit of lines segments each of whose lengths are bounded below by $2(\epsilon - \delta_i)$ (where δ_i is the Euclidean distance from p to q_i), we conclude L is a line segment of length at least 2ϵ . Note that so far, we have shown that every point in S_1 and every point in S_2 that fall in cases one and two are contained in an open line segment which is an integral curve of ν_H^\perp .

We are left with points of S_2 which fall into the third category. The collection of such points in S_2 is, by construction, closed and has empty interior. Thus, c_ϵ contains an open dense set of points that are either in S_1 or fall in one of the first two cases above. For each such points, Lemma 5.2

or the discussion of the first two cases yields an open line segment containing the point which is a subset of c_ϵ . Thus, as in the argument for case one, c_ϵ is piecewise linear and, by the smoothness of ν_H^\perp , must be a single straight line segment.

Using the arguments above for points in S_2 and Lemma 5.2 for points in S_1 , we see that integral curve of ν_H^\perp through any point contains a line segment through that point. Thus, all such integral curves are piecewise linear and, by the smoothness of ν_H^\perp , must be straight lines. Combining all of these arguments shows that Σ is foliated by straight line segments which are integral curves of ν_H^\perp . \square

Corollary 5.5. *Let S be a C^2 connected complete non-characteristic minimal surface without boundary in \mathbb{H}^1 . Then, S is foliated by horizontal straight lines which are integral curves of ν_H^\perp .*

Proof. Since S is assumed to have no boundary, for any $p \in S$ Lemma 5.4 implies that there exists an open neighborhood of p which is foliated by such straight line segments. By the smoothness of ν_H^\perp , we have that S itself is foliated by such straight line segments. It remains to show that the entirety of each line is contained in S .

Let $L : (-\epsilon, \epsilon) \rightarrow S$ be a line segment with $L(0) = p \in S$ and $L'(t) = \nu_H^\perp(L(t))$ and let $\tilde{L} : \mathbb{R} \rightarrow \mathbb{H}^1$ be the full line containing L so that $\tilde{L}(t) = L(t)$ for $t \in (-\epsilon, \epsilon)$. Let

$$I = \{t \in \mathbb{R} \mid \tilde{L}(t) \in S\}.$$

By construction, I is not empty since $0 \in I$. Let $t_i \in I$ be a sequence of parameters so that $t_i \rightarrow t_\infty$ where t_∞ is a limit point of I . By completeness of S , we must have that $\lim_{i \rightarrow \infty} \tilde{L}(t_i) = \tilde{L}(t_\infty)$ is an element of S . Thus, I is closed as it must contain all of its limit points. But, I is open as well. To see this, consider $p = \tilde{L}(t)$ for a fixed $t \in I$. As $\partial S = \emptyset$, p is in the interior of S and so, by Lemma 5.4, p is contained in a neighborhood which is foliated by straight lines which are integral curves of ν_H^\perp . Thus, I must contain an open neighborhood of t . Since I is both open and closed, we conclude that $I = \mathbb{R}$ and that $\tilde{L}(\mathbb{R}) \subset S$. \square

Proof of Theorem 5.1. By Corollary 5.5, we have that S is foliated by horizontal straight lines which are integral curves of ν_H^\perp . Let O be an open neighborhood of g_0 which may be written as a graph $(x, y, h(x, y))$ with $h \in C^2$. Consider a unit tangential vector field, \mathcal{W} , defined on O which is perpendicular (with respect to the fixed Riemannian metric) to ν_H^\perp . Let $(\gamma_1(s), \gamma_2(s), h_0(s))$ be an integral curve of \mathcal{W} so that $\gamma(0) = g_0$ with domain J . Note that $\gamma_1, \gamma_2, h_0 \in C^2(J)$ as ν_H^\perp is C^1 . Let N be the collection of lines in the foliation which pass through point of the curve $(\gamma_1(J), \gamma_2(J), h_0(J))$. Then, since for a fixed $s_0 \in J$, we have from (44)

$$\begin{aligned} \mathcal{L}'_{s_0}(r) &= (\gamma'_2(s_0), -\gamma'_1(s_0), -\frac{1}{2}(\gamma_1(s_0), \gamma_2(s_0)) \cdot (\gamma'_1(s_0), \gamma'_2(s_0))) \\ &= \gamma'_2(s_0) X_1 - \gamma'_1(s_0) X_2 = \nu_H^\perp, \end{aligned}$$

the line of the foliation passing through $(\gamma_1(s_0), \gamma_2(s_0), h_0(s_0))$ is given by

$$\mathcal{L}_{s_0}(r) = (\gamma_1(s_0) + r\gamma'_2(s_0), \gamma_2(s_0) - r\gamma'_1(s_0), h_0(s_0) - \frac{r}{2}(\gamma_1(s_0), \gamma_2(s_0)) \cdot (\gamma'_1(s_0), \gamma'_2(s_0)))$$

Thus, N may be parametrized by $\mathcal{L} : \mathbb{R} \times J \rightarrow \mathbb{H}^1$ given by

$$(48) \quad \mathcal{L}(r, s) = (\gamma_1(s) + r\gamma_2'(s), \gamma_2(s) - r\gamma_1'(s), h_0(s) - \frac{r}{2}\gamma(s) \cdot \gamma'(s)).$$

It remains to show that $\gamma = (\gamma_1, \gamma_2) \in C^3(J)$. As O is a graph over a region \bar{O} of the xy -plane, $\bar{\mathcal{L}}(r_0, s) = (\gamma_1(s) + r\gamma_2'(s), \gamma_2(s) - r\gamma_1'(s))$ parametrizes a subset of \bar{O} with $s \in J, r \in (-\epsilon, \epsilon)$ for ϵ sufficiently small. Under this parametrization, $V = \bar{p} \partial_x + \bar{q} \partial_y = \gamma_1'(s) \partial_x + \gamma_2'(s) \partial_y$. We first observe that, for a fixed $r = r_0$, the curve $s \rightarrow \bar{\mathcal{L}}(r_0, s)$ coincides with the integral curve of V through the point $\bar{\mathcal{L}}(r_0, 0)$ on their mutual domain of definition (we may assume, by shrinking J if necessary, that J is the mutual domain of definition). To see this, note that the definition of $\bar{\mathcal{L}}$ gives

$$\bar{\mathcal{L}}_s(r, s) = (\gamma_1'(s) + r\gamma_2''(s), \gamma_2'(s) - r\gamma_1''(s))$$

This implies

$$\langle \bar{\mathcal{L}}_s(r_0, s), V^\perp \rangle = \gamma_2'\gamma_1' + r\gamma_2'\gamma_2'' - \gamma_1'\gamma_2' + r\gamma_1''\gamma_1' = 0.$$

The last equality follows from the fact that $|\gamma'| \equiv 1$ on J . Let $\bar{c} \subset \mathbb{R}^2$ be the integral curve of V passing through $\bar{\mathcal{L}}(r_0, 0)$. We note that \bar{c} is parameterized by arc-length and, to avoid confusion, we will denote its parameter by ξ . Since V is C^1 , we have that $\bar{c} \in C^2(\xi)$. Moreover, since O is given by $(x, y, h(x, y))$ with $h \in C^2$, we see that $c(\xi) = h(\bar{c}(\xi))$ is $C^2(\xi)$ as well.

To facilitate our computations, we note that

$$|\bar{\mathcal{L}}_s(r_0, s)| = |1 - r_0\kappa(s)|.$$

This can be verified as follows. Recalling that $|\gamma'| = 1$ and that $\kappa = \gamma_1''\gamma_2' - \gamma_2''\gamma_1'$, one easily obtains

$$|\bar{\mathcal{L}}_s(r_0, s)|^2 = 1 - 2r\kappa(s) + r^2(\gamma_1''(s)^2 + \gamma_2''(s)^2).$$

Now, some elementary considerations give

$$\kappa(s)^2 = ((\gamma_1''(s)^2 + \gamma_2''(s)^2)|\gamma'(s)|^2 - 2(\gamma'(s) \cdot \gamma''(s))^2) = (\gamma_1''(s)^2 + \gamma_2''(s)^2),$$

and this implies the desired conclusion. Let now $\kappa_0 = \sup_{s \in J} |\kappa(s)|$. If $\kappa_0 = 0$, then γ is a line segment and hence γ is certainly C^3 . Assuming $\kappa_0 > 0$, we pick $r_0 < \min\{\kappa_0^{-1}, \epsilon\}$ which implies that $|\bar{\mathcal{L}}_s(r_0, s)| = |1 - r_0\kappa(s)| = 1 - r_0\kappa(s)$. We note that ξ is differentiable in s as $\bar{c}(\xi)$ is the reparameterization by arclength of $\bar{\mathcal{L}}(r_0, s)$ and that $\frac{d\xi}{ds} = 1 - r_0\kappa(s)$. Similarly,

$$\frac{ds}{d\xi} = \frac{1}{1 - r_0\kappa(s)}$$

which, by our choice of r_0 , is equal to $\sum_{n=0}^{\infty} (r_0 \kappa(s))^n$. Next, we compute

$$\begin{aligned}
 c'(\xi) &= \frac{d}{d\xi} h(\bar{c}(\xi)) = \frac{\partial}{\partial s} (h(\gamma_1(s) + r\gamma_2'(s), \gamma_2(s) - r\gamma_1'(s))) \frac{ds}{d\xi} \\
 &= \frac{\partial}{\partial s} \left(h_0(s) - \frac{r_0}{2} \gamma(s) \cdot \gamma'(s) \right) \frac{1}{1 - r_0 \kappa(s)} \\
 &= \left(h_0'(s) - \frac{r_0}{2} - \frac{r_0}{2} \gamma(s) \cdot \gamma''(s) \right) \frac{1}{1 - r_0 \kappa(s)} \\
 &= \left(h_0'(s) - \frac{r_0}{2} - \frac{r_0}{2} \gamma(s) \cdot \gamma''(s) \right) \left(\sum_{n=0}^{\infty} (r_0 \kappa(s))^n \right) \\
 &= h_0'(s) + r_0 \alpha(s) + r_0^2 \kappa(s) \alpha(s) + r_0^3 \kappa(s)^2 \alpha(s) + \dots
 \end{aligned}$$

where $\alpha(s) = -\frac{1}{2} - \frac{1}{2} \gamma(s) \cdot \gamma''(s) + \kappa(s) h_0'(s)$. At this point we can make some simplifications. First, we note that as $\kappa(s) = \gamma'' \cdot (\gamma')^\perp$, and $\gamma' \cdot \gamma'' = 0$ (as $|\gamma'(s)| = 1$), we have

$$\gamma''(s) = \kappa(s) (\gamma'(s))^\perp$$

So, letting $\beta(s) = -\frac{1}{2} \gamma \cdot (\gamma'(s))^\perp + h_0'(s)$, we rewrite $\alpha(s) = -\frac{1}{2} + \kappa(s) \beta(s)$. Moreover,

$$\begin{aligned}
 r_0 \alpha(s) + r_0^2 \kappa(s) \alpha(s) + r_0^3 \kappa(s)^2 \alpha(s) + \dots &= r_0 \alpha(s) \left(\sum_{n=0}^{\infty} (r_0 \kappa(s))^n \right) \\
 &= \frac{r_0 \alpha(s)}{1 - r_0 \kappa(s)} \\
 &= - \left(\frac{r_0}{2} \frac{1}{1 - r_0 \kappa(s)} - \beta(s) \frac{r_0 \kappa(s)}{1 - r_0 \kappa(s)} \right) \\
 &= - \left(\frac{r_0}{2} \frac{1}{1 - r_0 \kappa(s)} + \beta(s) - \frac{\beta}{1 - r_0 \kappa(s)} \right) \\
 &= - \left(\beta(s) + \frac{r_0 - 2\beta(s)}{1 - r_0 \kappa(s)} \right).
 \end{aligned}$$

We conclude that

$$c'(\xi) = h_0'(s) - \beta(s) - \frac{1}{2} \frac{r_0 - 2\beta(s)}{1 - r_0 \kappa(s)}$$

Since $c'(\xi)$ is again differentiable in ξ and ξ is differentiable in s , we conclude, by the chain rule, that $c'(\xi)$ is also differentiable in s . Noting that $h_0'(s)$ and $\beta(s)$ are once differentiable in s , we conclude that $(1 - r_0 \kappa(s))^{-1}$, and hence $\kappa(s)$, is differentiable in s . But, since $\gamma''(s) = \kappa(s) (\gamma'(s))^\perp$, $\gamma''(s)$ is differentiable and hence $\gamma \in C^3(s)$.

Lastly, we examine the impact of the assumption that S contains no characteristic points on the neighborhood N . Using the parametrization derived above, we see that the tangent space is spanned by ν_H^\perp and

$$\hat{W} = (\gamma_1'(s) + r\gamma_2''(s)) X_1 + (\gamma_2'(s) - r\gamma_1''(s)) X_2 + (W_0(s) - r + \frac{r^2}{2} \kappa(s)) T$$

where, as in the statement of the Theorem, we let $W_0(s) = h'_0(s) + \frac{1}{2}\gamma' \cdot \gamma^\perp$ and $\kappa(s) = \gamma'' \cdot (\gamma')^\perp$. S will have a characteristic point when $\langle \hat{W}, T \rangle = 0$, i.e. when $r = \frac{1 \pm \sqrt{1 - 2W_0(s)\kappa(s)}}{2W_0(s)}$. Thus, S is noncharacteristic if and only if $1 - 2W_0(s)\kappa(s) < 0$. \square

Note that, without loss of generality (by simply reparametrizing γ), we may assume that any fixed $s \in J$ may be treated as $s = 0$. We will use such a normalization and assume that J is a neighborhood of 0.

We wish to examine the behavior of this patch with respect to the notion of an X_1 graph. Consider the following definitions.

Definition 5.6. Let $C_1(x_0, y_0, t_0)$ denote the integral curve of the vector field X_1 passing through the point (x_0, y_0, t_0) . In other words,

$$C_1(x_0, y_0, t_0) = \left\{ \left(x_0 + r, y_0, t_0 - \frac{y_0}{2} r \right) \mid r \in \mathbb{R} \right\}.$$

Using Definition 5.6 we next introduce the notion of intrinsic projection of a point to the plane $x = 0$.

Definition 5.7. We define the intrinsic projection map

$$\Pi(x_0, y_0, t_0) = \{(0, y, t)\} \cap C_1(x_0, y_0, t_0) = (0, y_0, t_0 + y_0 x_0 / 2).$$

The following equation follows directly from the definition.

$$(49) \quad \Pi \circ \mathcal{L}(r, s) = (0, \gamma_2(s) - r\gamma'_1(s), h_0(s) + \frac{1}{2}\gamma_1(s)\gamma_2(s) - r\gamma_1(s)\gamma'_1(s) - \frac{r^2}{2}\gamma'_1(s)\gamma'_2(s))$$

Lemma 5.8. Let S be a portion of an H -minimal surface parameterized by a seed curve/height function pair $(\gamma(s), h_0(s))$ via (39) with $r \in \mathbb{R}$, $s \in I$. Let $P(s, r) = \Pi \circ \mathcal{L}(r, s)$ be given as in (49). There exists an interval $J \subset I$ containing so that $P : \mathbb{R} \times J \subset \mathbb{R}^2_{(r,s)} \rightarrow \mathbb{R}^2_{(y,t)}$ is a one-to-one C^2 diffeomorphism onto its image.

Proof. The following properties of the seed curve $\gamma : I \rightarrow \mathbb{R}^2$ are essential to our proof. We gather them here for the sake of convenience.

- (i) $|\gamma'(s)| = 1$.
- (ii) $1 - 2W_0(s)\kappa(s) < 0$.
- (iii) There exists an interval $J \subset I$ such that for all $s \in J$, $\gamma'_1(s) \neq 0$.

Properties (i), (ii) and the definitions of W_0 and κ were establish in Theorem 5.1. Suppose (iii) is not true, then together with (i) we would have $\gamma'(s) = (0, 1)1$ for all $s \in I$. This would implies $\kappa(s) = \gamma''(s) \cdot \gamma'(s)^\perp$ vanishes identically on I and hence (ii) would not be possible. Therefore, by the continuity of γ'_1 , we can extract a sub-interval J of I on which $\gamma'_1(s) \neq 0$. To continue we define two auxiliary functions ζ and Ψ by means of γ as follows.

$$\begin{aligned} \zeta : \mathbb{R} \times J &\rightarrow \mathbb{R}^2, & \zeta(r, s) &= (\gamma_2(s) - r\gamma'_1(s), s), \\ \Psi : \zeta(\mathbb{R} \times J) &\rightarrow \mathbb{R}^2, & (u, v) = \Psi(u, s) &= \left(u, \sigma(s) + F(s)u + \frac{G(s)}{2}u^2 \right). \end{aligned}$$

where $F, G, \sigma : J \rightarrow \mathbb{R}$ is given by

$$(50) \quad \begin{aligned} F(s) &= \gamma_1(s) + \frac{\gamma_2(s)\gamma_2'(s)}{\gamma_1'(s)} = \frac{\gamma \cdot \gamma'}{\gamma_1'} \\ G(s) &= -\frac{\gamma_2'(s)}{\gamma_1'(s)} \\ \sigma(s) &= h_0(s) - \frac{1}{2}\gamma_2(s)F(s). \end{aligned}$$

Due to property (iii) above and the fact that $\gamma \in C^3(I)$, the functions $\zeta, \Psi, F, G, \sigma$ are well defined and are $C^2(J)$. One can verify by a straight forward computation that

$$\Pi \circ \mathcal{L}(r, s) = \Psi \circ \zeta(r, s).$$

Therefore, if we show that $\Psi \circ \zeta : \mathbb{R} \times J \rightarrow \mathbb{R}^2$ is one one then $\Pi \circ \mathcal{L}$ is also one one. To this end, we will show separately that both ζ and Ψ are one to one. The fact that ζ is one one is easy to verify and follows from the fact that $\gamma_1'(s) \neq 0$ on J . We also note that

$$\zeta(\mathbb{R} \times J) = \mathbb{R} \times J.$$

To show that Ψ is one to one, we first consider its second component: $v(u, s) = \sigma(s) + F(s)u + \frac{G(s)}{2}u^2$. We have

$$\frac{\partial}{\partial s}v(u, s) = \sigma'(s) + F'(s)u + \frac{G'(s)}{2}u^2.$$

Although it is tedious, nevertheless one can verify by straight forward computations that the following identity holds for any $s \in J$ and any $u \in \mathbb{R}$:

$$F'(s)^2 - 2\sigma'(s)G'(s) = 1 - 2W_0(s)\kappa(s) + (|\gamma'(s)|^2 + 1)(|\gamma'(s)|^2 - 1) < 0.$$

The strict inequality above is due to properties (i) and (ii) of γ . This in turn implies that the quadratic expression in u

$$\frac{\partial}{\partial s}v(u, s) = \sigma'(s) + F'(s)u + \frac{G'(s)}{2}u^2$$

do not vanish for any fixed $u \in \mathbb{R}$ and any $s \in J$. Hence we have

$$\left| \frac{\partial}{\partial s}v(u, s) \right| > 0, \quad s \in J$$

that is, $v(u, s)$ is monotone in s for any fixed $u \in \mathbb{R}$. We infer from this fact and the definition of Ψ that Ψ is one one. This completes the proof. \square

Several important facts about the functions F, G, σ, Ψ were established in the proof of Lemma 5.8 we single them out here for references.

Proposition 5.9. *The functions F, G, σ satisfy*

$$(51) \quad F'(s)^2 - 2\sigma'(s)G'(s) < 0.$$

The function $\Psi : \mathbb{R} \times J \rightarrow \mathbb{R}^2$ is invertible on its image. We let $(u, s) = \Psi^{-1}(u, v)$. In particular, $s = s(u, v)$ is the second component of Ψ^{-1} .

These two lemmas show that every C^2 noncharacteristic complete noncompact embedded H -minimal surface which is not itself a vertical plane contains a subsurface which can be written as an intrinsic graph. To make the presentation as clean as possible, we prove an intermediate lemma.

Lemma 5.10. *Let S be a C^2 noncharacteristic complete noncompact embedded H -minimal surface which is not itself a vertical plane and let J and the functions F, G, σ, Ψ be the ones from the proof of Lemma 5.8 and s as in Proposition 5.9. If $\phi : \Psi(\mathbb{R} \times J) \rightarrow \mathbb{R}^2$ is given by*

$$\phi(u, v) = F(s(u, v)) + uG(s(u, v)) \quad \text{for } (u, v) \in \Omega = \Psi(\mathbb{R} \times J).$$

Then

$$S_0 = \{(0, u, v) \circ (\phi(u, v), 0, 0) \mid (u, v) \in \Omega\}$$

is a sub surface of S .

Proof. With the functions $\Psi, \phi, s, F, G, \sigma$ and Ω as in the statement of the Lemma, we define $\Phi : \Omega \rightarrow \mathbb{H}^1$ as follows

$$\Phi(u, v) = \left(\phi(u, v), u, v - \frac{1}{2} u \phi(u, v) \right).$$

Our intention is to show that $\Phi(\Omega) = \mathcal{L}(\mathbb{R} \times J)$. We begin by comparing the second components of Φ and \mathcal{L} . Note that if

$$(52) \quad u = \gamma_2(s) - r \gamma_1'(s),$$

then

$$\begin{aligned} (53) \quad \phi(u, v) &= F(s(u, v)) + u G(s(u, v)) \\ &= F(s) + (\gamma_2(s) - r \gamma_1'(s)) G(s) \\ (\text{by (50)}) &= \gamma_1(s) + \frac{\gamma_2(s) \gamma_2'(s)}{\gamma_1'(s)} - \left(\gamma_2(s) - r \gamma_1'(s) \right) \frac{\gamma_2'(s)}{\gamma_1'(s)} \\ &= \frac{\gamma_1(s) \gamma_1'(s) + \gamma_2(s) \gamma_2'(s) - \gamma_2(s) \gamma_2'(s) + r \gamma_1'(s) \gamma_2'(s)}{\gamma_1'(s)} \\ &= \gamma_1(s) + r \gamma_2'(s), \end{aligned}$$

which is the first component of \mathcal{L} . We now turn to the third component of Φ . Keeping in mind that for $(u, v) \in \Omega = \Psi(\mathbb{R} \times J)$ we have

$$v = \sigma(s) + F(s)u + \frac{G(s)}{2} u^2$$

hence

$$\begin{aligned}
v - \frac{1}{2} u \phi(u, v) &= \sigma(s) + F(s)u + \frac{G(s)}{2} u^2 - \frac{1}{2} u \phi(u, v) \\
(\text{by (52), (50) and (53)}) &= h_0(s) - \frac{1}{2} \gamma_2(s) \left(\gamma_1(s) + \frac{\gamma_2(s) \gamma_2'(s)}{\gamma_1'(s)} \right) \\
&\quad + \left(\gamma_1(s) + \frac{\gamma_2(s) \gamma_2'(s)}{\gamma_1'(s)} \right) (\gamma_2(s) - r \gamma_1'(s)) \\
&\quad - \frac{1}{2} \frac{\gamma_2'(s)}{\gamma_1'(s)} (\gamma_2(s) - r \gamma_1'(s))^2 - \frac{1}{2} (\gamma_2(s) - r \gamma_1'(s)) (\gamma_1(s) + r \gamma_2'(s)) \\
&= h_0(s) - \frac{r}{2} \gamma(s) \cdot \gamma'(s)
\end{aligned}$$

which is the third component of \mathcal{L} . □

Finally, we turn to the

Proof of Theorem B. Since S is not itself a vertical plane, Lemma 5.3 guarantee the existence of a point $g_o \in S$ and a neighborhood N of g_o such that N can be written as a graph over the plane $t = 0$. Theorem 5.1 then provides the necessary parameterization of such a neighborhood by the map \mathcal{L} whose domain is $\mathbb{R} \times J$. Lemmas 5.8, 5.10 and Proposition 5.9 then show that the portion $\mathcal{L}(\mathbb{R} \times J) \subset S$ can be reparameterized to conform to Definition 1.9 hence, establishing the required δ -graphical strip. □

Combining this with Theorem B, we can now easily prove the main Theorem.

Proof of Theorem C. Suppose S is a C^2 complete embedded noncharacteristic H -minimal surface without boundary which is not a vertical plane. Then, Theorem B shows that S contains an intrinsic graphical strip, S_0 , and thus, by Theorem A, S_0 , and hence S , is not stable. □

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907

E-mail address, Donatella Danielli: danielli@math.purdue.edu

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907

E-mail address, Nicola Garofalo: garofalo@math.purdue.edu

DEPARTMENT OF MATHEMATICS, SAN DIEGO CHRISTIAN COLLEGE, SAN DIEGO, CA 92019

E-mail address, Duy-Minh Nhieu: dnhieu@sdcc.edu

DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NH 03755

E-mail address: scott.pauls@dartmouth.edu